The Linear Tower of Hanoi

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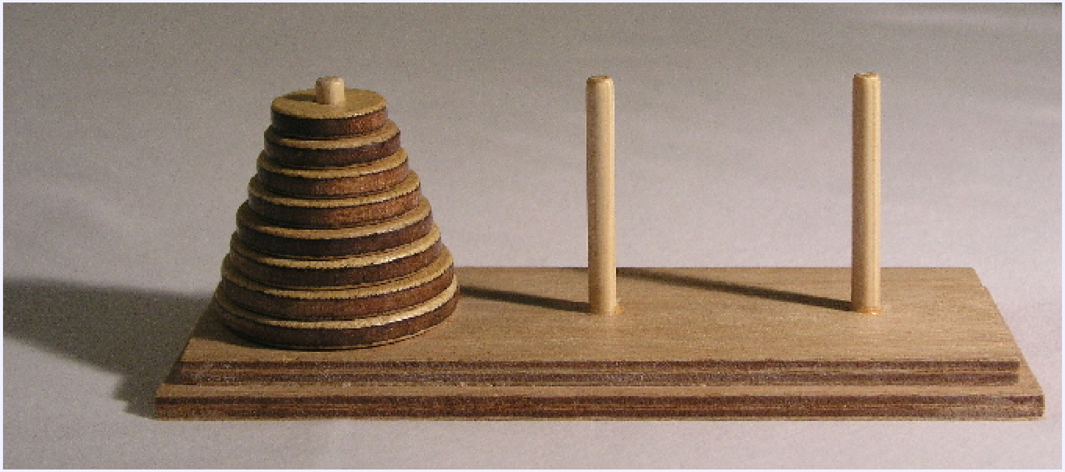
Lisboa (Portugal), 2019–01–29



The Tower of Hanoi with three pegs in a row

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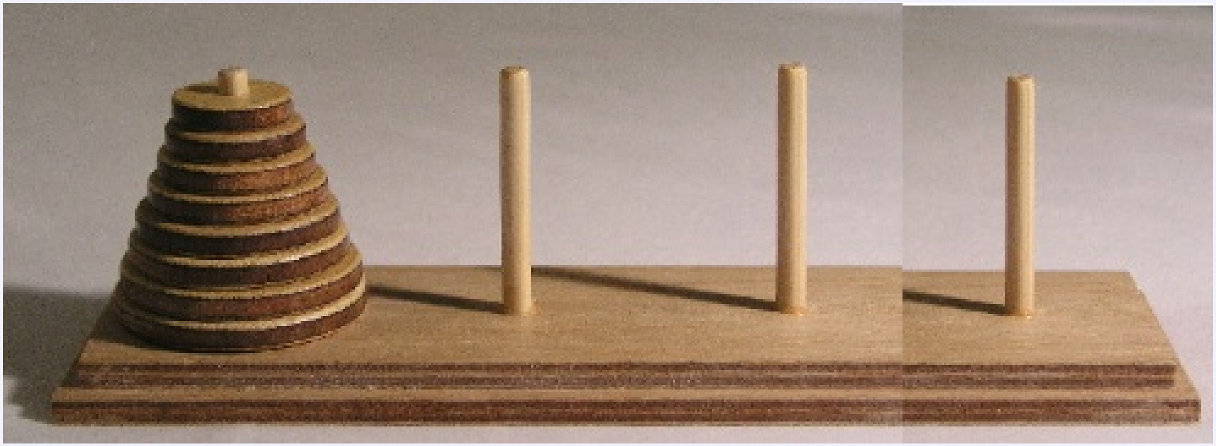


The Tower of Hanoi with four pegs in a row

The problem

Consider the Tower of Hanoi with four (or more) vertical pegs arranged in a horizontal row.

What is the number of moves necessary (and sufficient) to transfer n discs from the peg on one end of the row to the peg on the other end if in addition to the classical rules discs may only be moved to a neighboring peg?

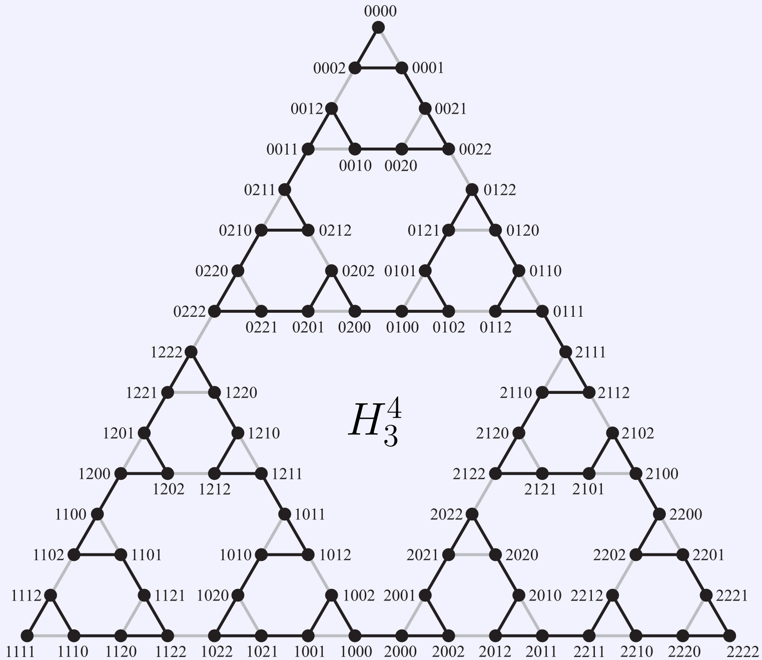
Construct an efficient algorithm to find a(ll) shortest solution(s)!

Hint: the sequence of move numbers starts

$$0, 3, 10, 19, 34, 57, 88, 123, 176, 253, 342, 449, 572, 749, 980, 1261,$$
$$1560, 1903, 2328, 2889, 3562, \dots$$

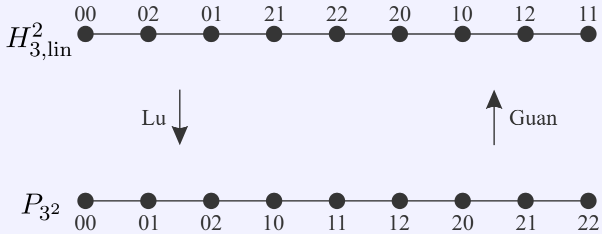
(B. Lužar, 2019)

Mathematical background

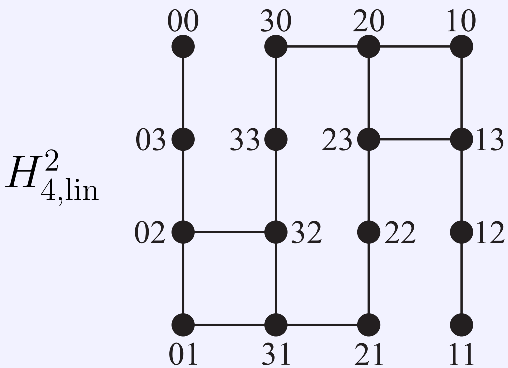


solution of $0^n \to 1^n$ in H_3^n is a path $\cong R^n$ of length $2^n - 1$

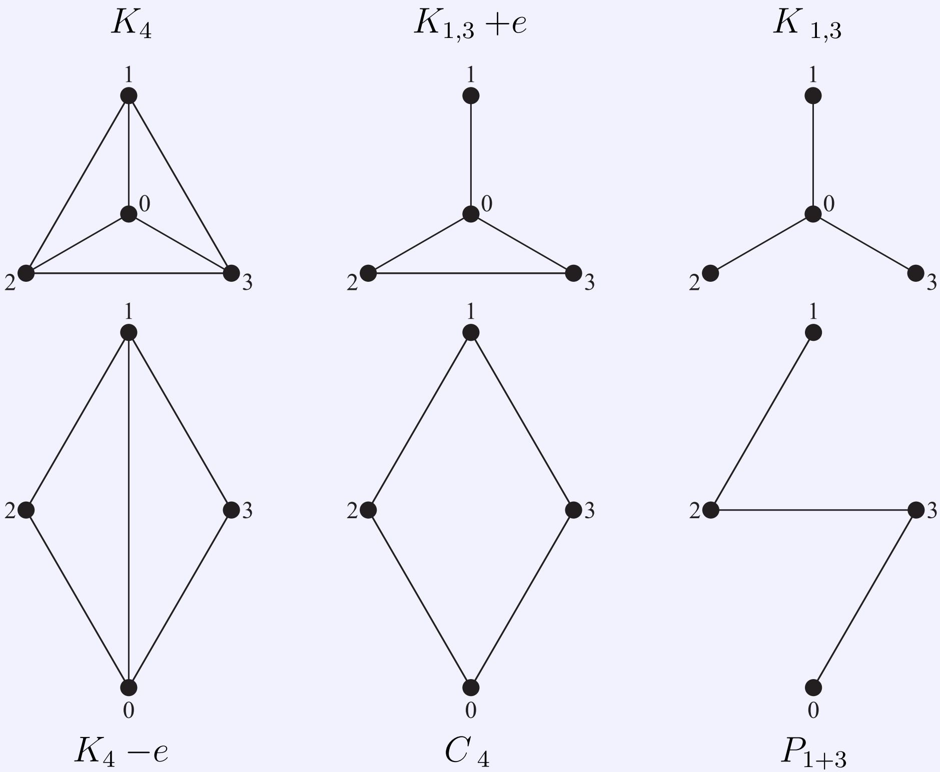
solution of $0^n \to 1^n$ in $H_{3,\text{lin}}^n$ is a (hamiltonian) path in H_3^n of length $3^n - 1$



In particular, there is a unique optimal solution. This is *not* so for the linear case with $H^1_{4,\text{lin}} \cong P_4$, where for $H^2_{4,\text{lin}}$ there are 4 shortest paths (of length 10) from 00 to 11:



The Dudeney-Stockmeyer Conjecture



does *not* apply for $H^1_{4,\mathrm{lin}} \cong P_{1+3}$ and task $0^n \to 1^n$.

Theorem *In a shortest solution in $H^{1+n}_{p,\mathrm{lin}}$, $n \in \mathbb{N}_0$, $p \in \mathbb{N}_3$, the largest disc always moves towards its goal peg, i.e. exactly $\lambda \in [p]_0$ times, if the goal peg is λ steps away from the initial peg in $H^1_{p,\mathrm{lin}} \cong P_p$.*

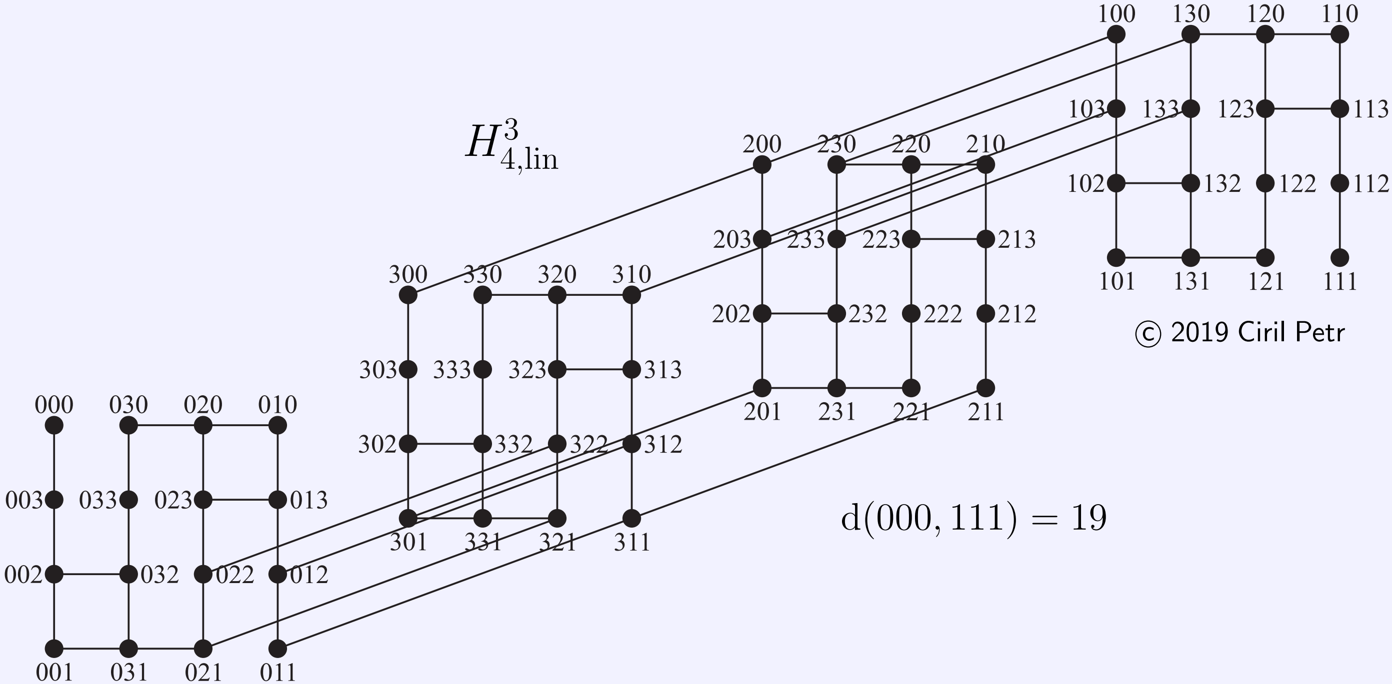
Moreover, for each move of disc $n+1$ there are $(p-2)^n$ edges available for a (possibly not optimal) solution.

Proof. If disc $n+1$ were to move "backwards" from some peg, it would have to pass that peg again eventually, so one could have left it there anyway.

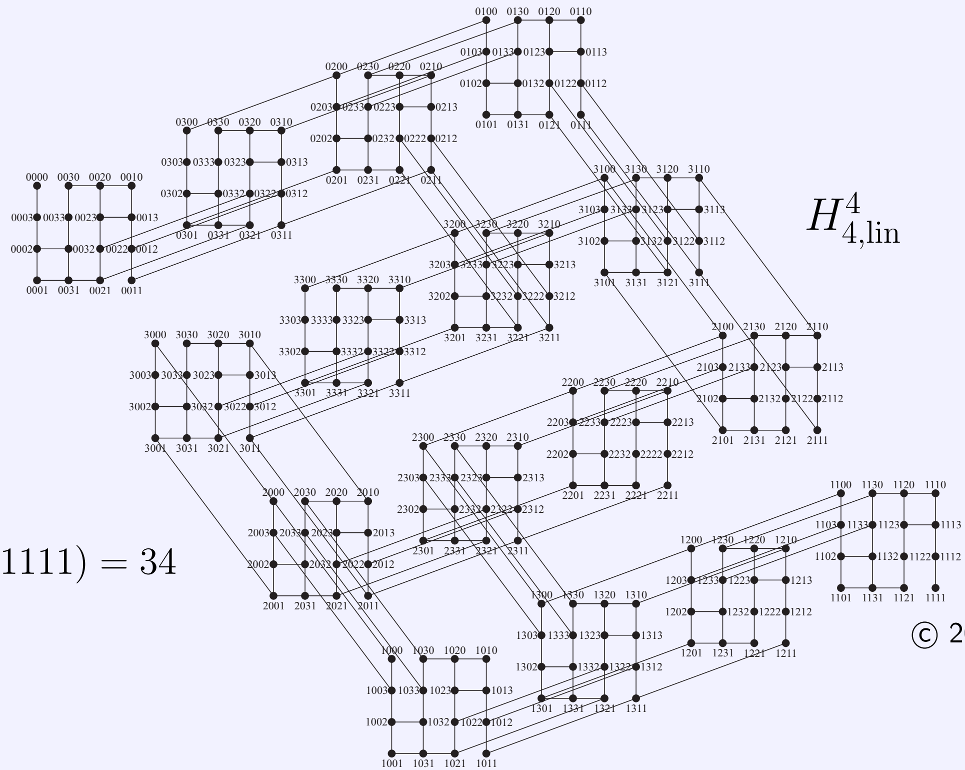
The second statement follows from the rules of the game. □

$$\|H^{1+n}_{p,\mathrm{lin}}\| = p\|H^n_{p,\mathrm{lin}}\| + (p-1)(p-2)^n$$

$$|H^n_{p,\mathrm{lin}}| = p^n,\ \|H^n_{p,\mathrm{lin}}\| = \tfrac{1}{2}(p-1)\left(p^n - (p-2)^n\right) \overset{p=4}{=} 3 \cdot 2^{n-1}(2^n - 1)$$

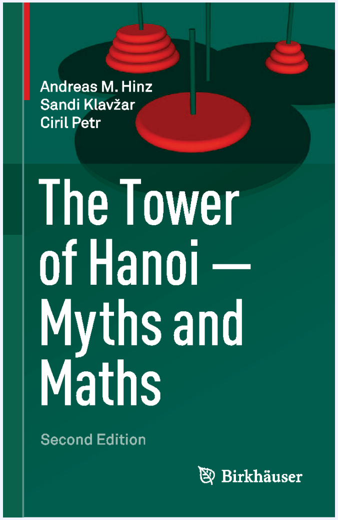

$H^3_{4,\mathrm{lin}}$
100 130 120 110
103 133 123 113
102 132 122 112
101 131 121 111
200 230 220 210
203 233 223 213
202 232 222 212
201 231 221 211
300 330 320 310
303 333 323 313
302 332 322 312
301 331 321 311
000 030 020 010
003 033 023 013
002 032 022 012
001 031 021 011
$\mathrm{d}(000, 111) = 19$

$$H^4_{4,\mathrm{lin}}$$



$$\mathrm{d}(0000, 1111) = 34$$

Further reading:



Cham, 2018

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