[12pt]beamer

Splitting the Integers by Sequences

Aviezri S. Fraenkel
Weizmann Institute of Science
Rehovot, Israel

January 31, 2017

Arithmetic Sequences

Arithmetic Sequences

- The arithmetic sequences $\{2n - 1 : n \geq 1\}$, $\{2n : n \geq 1\}$ partition the positive integers $\mathbb{Z}_{\geq 1}$.

Arithmetic Sequences

- The arithmetic sequences $\{2n - 1 : n \geq 1\}$, $\{2n : n \geq 1\}$ partition the positive integers $\mathbb{Z}_{\geq 1}$.
- So do $\{4n - 3 : n \geq 1\}$, $\{4n - 1 : n \geq 1\}$, $\{2n : n \geq 1\}$.

Arithmetic Sequences

- The arithmetic sequences $\{2n-1 : n \geq 1\}$, $\{2n : n \geq 1\}$ partition the positive integers $\mathbb{Z}_{\geq 1}$.
- So do $\{4n-3 : n \geq 1\}$, $\{4n-1 : n \geq 1\}$, $\{2n : n \geq 1\}$.
- Notice that for both systems, the two largest **moduli** $(2, 2$ and $4, 4)$ are identical.

Arithmetic Sequences

- The arithmetic sequences $\{2n-1 : n \geq 1\}$, $\{2n : n \geq 1\}$ partition the positive integers $\mathbb{Z}_{\geq 1}$.
- So do $\{4n-3 : n \geq 1\}$, $\{4n-1 : n \geq 1\}$, $\{2n : n \geq 1\}$.
- Notice that for both systems, the two largest **moduli** ($2, 2$ and $4, 4$) are identical.
- Davenport, Mirsky, D. Newman, Radó proved, using a slick generating function and complex root of unity proof, that in any partitioning of $\mathbb{Z}_{\geq 1}$ into $m \geq 2$ arithmetic sequences, the two largest moduli are identical.

Theorem Mirsky et al

Theorem

If the integer system $\bigcup\{na_i + b_i\}_{i=1}^{m}$ *is complementary,* $a_1 \leq a_2 \leq \ldots \leq a_m$ *and* $m \geq 2$*, then* $a_{m-1} = a_m$*.*

Theorem Mirsky et al

Theorem

If the integer system $\bigcup\{na_i + b_i\}_{i=1}^{m}$ *is complementary,* $a_1 \leq a_2 \leq \ldots \leq a_m$ *and* $m \geq 2$, *then* $a_{m-1} = a_m$.

- **Proof**
- Generating function statement of theorem's hypothesis: $\sum_{i=1}^{m} z^{b_i}/(1 - z^{a_i}) = z/(1 - z)$.

Theorem Mirsky et al

Theorem

If the integer system $\bigcup\{na_i + b_i\}_{i=1}^{m}$ *is complementary,* $a_1 \leq a_2 \leq \ldots \leq a_m$ *and* $m \geq 2$*, then* $a_{m-1} = a_m$*.*

- **Proof**
- Generating function statement of theorem's hypothesis: $\sum_{i=1}^{m} z^{b_i}/(1 - z^{a_i}) = z/(1 - z)$.
- Let ρ be a primitive a_mth root of unity: $\rho^{a_m} = 1$, and no smaller power of ρ gives 1.

Theorem Mirsky et al

Theorem

If the integer system $\bigcup\{na_i + b_i\}_{i=1}^m$ *is complementary,* $a_1 \le a_2 \le \ldots \le a_m$ *and* $m \ge 2$*, then* $a_{m-1} = a_m$*.*

- **Proof**
- Generating function statement of theorem's hypothesis: $\sum_{i=1}^m z^{b_i}/(1 - z^{a_i}) = z/(1-z)$.
- Let ρ be a primitive a_mth root of unity: $\rho^{a_m} = 1$, and no smaller power of ρ gives 1.
- Suppose $a_{m-1} < a_m$. Let $z \to \rho$. Then $z^{b_m}/(1 - z^{a_m})$ is the only term growing unboundedly. Contradiction.

Theorem

If the integer system $\bigcup\{na_i + b_i\}_{i=1}^{m}$ *is complementary,* $a_1 \leq a_2 \leq \ldots \leq a_m$ *and* $m \geq 2$*, then* $a_{m-1} = a_m$.

- **Proof**
- Generating function statement of theorem's hypothesis: $\sum_{i=1}^{m} z^{b_i}/(1 - z^{a_i}) = z/(1 - z)$.
- Let ρ be a primitive a_mth root of unity: $\rho^{a_m} = 1$, and no smaller power of ρ gives 1.
- Suppose $a_{m-1} < a_m$. Let $z \to \rho$. Then $z^{b_m}/(1 - z^{a_m})$ is the only term growing unboundedly. Contradiction.
- ... Erdos ... Berger, Felzenbaum, F. ...independently by Simpson.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ *irrational*, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ *partition* $\mathbb{Z}_{\geq 1}$.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ *irrational*, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ *partition* $\mathbb{Z}_{\geq 1}$.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ *irrational*, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ *partition* $\mathbb{Z}_{\geq 1}$.

- This is an irrational partitioning system with two *distinct* moduli.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ *irrational*, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ *partition* $\mathbb{Z}_{\geq 1}$.

- This is an irrational partitioning system with two *distinct* moduli.
- In Feb 1973 I showed that for every $m \geq 3$, the rational system $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n\geq 1}$ with $\alpha_i = (2^m - 1)/2^{m-i}$, $\beta_i = -2^{i-1} + 1$, $i = 1, \ldots, m$ partitions $\mathbb{Z}_{\geq 1}$.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ *irrational*, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ *partition* $\mathbb{Z}_{\geq 1}$.

- This is an irrational partitioning system with two *distinct* moduli.
- In Feb 1973 I showed that for every $m \geq 3$, the rational system $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n\geq 1}$ with $\alpha_i = (2^m - 1)/2^{m-i}$, $\beta_i = -2^{i-1} + 1$, $i = 1, \ldots, m$ partitions $\mathbb{Z}_{\geq 1}$.
- This is a partitioning system with $m \geq 3$ **distinct** moduli.

Non-integer moduli

Theorem

$\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1 \implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ partition $\mathbb{Z}_{\geq 1}$.

- This is an irrational partitioning system with two *distinct* moduli.
- In Feb 1973 I showed that for every $m \geq 3$, the rational system $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n\geq 1}$ with $\alpha_i = (2^m - 1)/2^{m-i}$, $\beta_i = -2^{i-1} + 1$, $i = 1, \ldots, m$ partitions $\mathbb{Z}_{\geq 1}$.
- This is a partitioning system with $m \geq 3$ **distinct** moduli.
- Example: $m = 3$.

n	$\lfloor 7n/4 \rfloor$	$\lfloor 7n/2 \rfloor - 1$	$7n - 3$
1	1	2	4
2	3	6	
3	5		
4	7		

Conjecture

Conjecture

- I further conjectured there:

Conjecture

- I further conjectured there:
- Every partition of $\mathbb{Z}_{\geq 1}$ into $m \geq 3$ sets $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n \in \mathbb{Z}_{\geq 1}}$ with α_i, β_i real, $\alpha_k \neq \alpha_\ell$ for all $k \neq \ell$ satisfies $\alpha_i = (2^m - 1)/2^{m-i}$, $i = 1, \ldots, m$.

Conjecture

- I further conjectured there:
- Every partition of $\mathbb{Z}_{\geq 1}$ into $m \geq 3$ sets $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n \in \mathbb{Z}_{\geq 1}}$ with α_i, β_i real, $\alpha_k \neq \alpha_\ell$ for all $k \neq \ell$ satisfies $\alpha_i = (2^m - 1)/2^{m-i}$, $i = 1, \ldots, m$.
- In other words, the **only** partitioning system by sequences into $m \geq 3$ sets with **distinct** moduli is the indicated rational system!

Conjecture

- I further conjectured there:
- Every partition of $\mathbb{Z}_{\geq 1}$ into $m \geq 3$ sets $\{\lfloor n\alpha_i + \beta_i \rfloor\}_{n \in \mathbb{Z}_{\geq 1}}$ with α_i, β_i real, $\alpha_k \neq \alpha_\ell$ for all $k \neq \ell$ satisfies $\alpha_i = (2^m - 1)/2^{m-i}$, $i = 1, \ldots, m$.
- In other words, the **only** partitioning system by sequences into $m \geq 3$ sets with **distinct** moduli is the indicated rational system!
- Google 'F Conjecture'.

Back to the irrational case

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies \{\lfloor n\alpha \rfloor\}_{n \geq 1}$, $\{\lfloor n\beta \rfloor\}_{n \geq 1}$ partition $\mathbb{Z}_{\geq 1}$.

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies \{\lfloor n\alpha \rfloor\}_{n \geq 1}$, $\{\lfloor n\beta \rfloor\}_{n \geq 1}$ partition $\mathbb{Z}_{\geq 1}$.
- Ron Graham noticed that in any such system, we can also split the multiplying n into arithmetic sequences to get a multiple-set partitioning.

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ partition $\mathbb{Z}_{\geq 1}$.
- Ron Graham noticed that in any such system, we can also split the multiplying n into arithmetic sequences to get a multiple-set partitioning.
- Example: $\{\lfloor (2n)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor (2n-1)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ also partition $\mathbb{Z}_{\geq 1}$.

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ partition $\mathbb{Z}_{\geq 1}$.
- Ron Graham noticed that in any such system, we can also split the multiplying n into arithmetic sequences to get a multiple-set partitioning.
- Example: $\{\lfloor (2n)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor (2n-1)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ also partition $\mathbb{Z}_{\geq 1}$.
- This can be re-written in the form: $\{\lfloor n(2\alpha) \rfloor\}_{n\geq 1}$, $\{\lfloor n(2\alpha) - \alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$. So here we have again a splitting with two identical moduli: 2α.

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies$ $\{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ partition $\mathbb{Z}_{\geq 1}$.
- Ron Graham noticed that in any such system, we can also split the multiplying n into arithmetic sequences to get a multiple-set partitioning.
- Example: $\{\lfloor (2n)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor (2n-1)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ also partition $\mathbb{Z}_{\geq 1}$.
- This can be re-written in the form: $\{\lfloor n(2\alpha) \rfloor\}_{n\geq 1}$, $\{\lfloor n(2\alpha) - \alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$. So here we have again a splitting with two identical moduli: 2α.
- Of course also the multiplier n of β can be split into arbitrary arithmetic sequences.

Back to the irrational case

- As we saw above, $\alpha > 0$, $\beta > 0$ irrational, $1/\alpha + 1/\beta = 1$ $\implies \{\lfloor n\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ partition $\mathbb{Z}_{\geq 1}$.
- Ron Graham noticed that in any such system, we can also split the multiplying n into arithmetic sequences to get a multiple-set partitioning.
- Example: $\{\lfloor (2n)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor (2n-1)\alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$ also partition $\mathbb{Z}_{\geq 1}$.
- This can be re-written in the form: $\{\lfloor n(2\alpha) \rfloor\}_{n\geq 1}$, $\{\lfloor n(2\alpha) - \alpha \rfloor\}_{n\geq 1}$, $\{\lfloor n\beta \rfloor\}_{n\geq 1}$. So here we have again a splitting with two identical moduli: 2α.
- Of course also the multiplier n of β can be split into arbitrary arithmetic sequences.
- It follows from the result on partitioning with arithmetic sequences, that in any such irrational system with $m \geq 3$, two moduli (not necessarily the largest ones) are identical.

The most far-reaching advance on the F-Conjecture to date

The most far-reaching advance on the F-Conjecture to date

- Ron Graham then proceeded to prove that these types of irrational partitioning systems are the only ones that can exist!

The most far-reaching advance on the F-Conjecture to date

- Ron Graham then proceeded to prove that these types of irrational partitioning systems are the only ones that can exist!
- Interim conclusion: The F-Conjecture is proved for the integers, proved for the irrationals,

  but is wide open for the rationals.

The most far-reaching advance on the F-Conjecture to date

- Ron Graham then proceeded to prove that these types of irrational partitioning systems are the only ones that can exist!
- Interim conclusion: The F-Conjecture is proved for the integers, proved for the irrationals,

  but is wide open for the rationals.
- I find this to be the most tantalizing and fascinating aspect of the F-conjecture.

Other advances on the conjecture

Other advances on the conjecture

- Some advances made in my initial paper (1973), in papers by Simpson, in papers by Morikawa.

Other advances on the conjecture

- Some advances made in my initial paper (1973), in papers by Simpson, in papers by Morikawa.
- The F-Conjecture was proved by Simpson if the smallest modulus is at most $3/2$, by Morikawa for $m = 3$ and, under some condition, for $m = 4$. Proofs in terms of balanced sequences have been given for $m = 3$ by Tijdeman and for $m = 4$ by Altman, Gaujal and Hordijk (unconditional). Later it was proved by Tijdeman for $m = 5$ and 6, by Barát and Varjú for $m = 7$.

Other advances on the conjecture

- Some advances made in my initial paper (1973), in papers by Simpson, in papers by Morikawa.
- The F-Conjecture was proved by Simpson if the smallest modulus is at most $3/2$, by Morikawa for $m = 3$ and, under some condition, for $m = 4$. Proofs in terms of balanced sequences have been given for $m = 3$ by Tijdeman and for $m = 4$ by Altman, Gaujal and Hordijk (unconditional). Later it was proved by Tijdeman for $m = 5$ and 6, by Barát and Varjú for $m = 7$.
- Morikawa gave necessary and sufficient conditions for two rational sequences to be disjoint. Simpson simplified his difficult proof and dubbed it 'Japanese Remainder Theorem' in honor of Morikawa.

An application: 'Just-In-Time' systems

An application: 'Just-In-Time' systems

- 'Just-In-Time' systems originated at Toyota automotive, but have by now become standard at any large-scale corporation.

An application: 'Just-In-Time' systems

- 'Just-In-Time' systems originated at Toyota automotive, but have by now become standard at any large-scale corporation.
- Toyota noticed that they spend a huge amount of resources in maintaining inventories of automotive parts, some of which become obsolete even before being called into use.

An application: 'Just-In-Time' systems

- 'Just-In-Time' systems originated at Toyota automotive, but have by now become standard at any large-scale corporation.
- Toyota noticed that they spend a huge amount of resources in maintaining inventories of automotive parts, some of which become obsolete even before being called into use.
- They sought methods to guarantee that parts needed for manufacturing arrive *just in time*!

An application: 'Just-In-Time' systems

- 'Just-In-Time' systems originated at Toyota automotive, but have by now become standard at any large-scale corporation.
- Toyota noticed that they spend a huge amount of resources in maintaining inventories of automotive parts, some of which become obsolete even before being called into use.
- They sought methods to guarantee that parts needed for manufacturing arrive *just in time*!
- I think Tijdeman was the first to connect the F-Conjecture (partitioning numbers) with modern 'Just-In-Time' systems (partitioning time).

Academic 'Just-In-Time' research; Games

Academic 'Just-In-Time' research; Games

- Kubiak; Kubiak and Sethi used 'Just-In-Time' methods to shed new light on the F-Conjecture.

Academic 'Just-In-Time' research; Games

- Kubiak; Kubiak and Sethi used 'Just-In-Time' methods to shed new light on the F-Conjecture.
- So did Steiner, Brauner and Crama.

Academic 'Just-In-Time' research; Games

- Kubiak; Kubiak and Sethi used 'Just-In-Time' methods to shed new light on the F-Conjecture.
- So did Steiner, Brauner and Crama.
- I was once in Liège at a conference organized by Michel Rigo on combinatorics on words, when Crama and Brauner intercepted me to listen to them re F-Conjecture and 'Just-In-Time' systems.

Academic 'Just-In-Time' research; Games

- Kubiak; Kubiak and Sethi used 'Just-In-Time' methods to shed new light on the F-Conjecture.
- So did Steiner, Brauner and Crama.
- I was once in Liège at a conference organized by Michel Rigo on combinatorics on words, when Crama and Brauner intercepted me to listen to them re F-Conjecture and 'Just-In-Time' systems.
- The conjecture also induced the mouse game, rat game, fat rat game with 2, 3, 4 piles respectively, whose P-positions are the cases $m = 2, 3, 4$ of the conjecture respectively.

Academic 'Just-In-Time' research; Games

- Kubiak; Kubiak and Sethi used 'Just-In-Time' methods to shed new light on the F-Conjecture.
- So did Steiner, Brauner and Crama.
- I was once in Liège at a conference organized by Michel Rigo on combinatorics on words, when Crama and Brauner intercepted me to listen to them re F-Conjecture and 'Just-In-Time' systems.
- The conjecture also induced the mouse game, rat game, fat rat game with 2, 3, 4 piles respectively, whose P-positions are the cases $m = 2, 3, 4$ of the conjecture respectively.
- For $2, 3$ we could formulate game rules, but for $m = 4$ no game rules were found.

After 44 years, why hasn't the F-Conjecture been settled?

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.
- We did not manage to settle the conjecture, but worked jointly fruitfully on related problems, especially with Jamie Simpson.

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.
- We did not manage to settle the conjecture, but worked jointly fruitfully on related problems, especially with Jamie Simpson.
- The group then got an extension of their grant and they again invited me, with the same type of results.

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.
- We did not manage to settle the conjecture, but worked jointly fruitfully on related problems, especially with Jamie Simpson.
- The group then got an extension of their grant and they again invited me, with the same type of results.
- After I returned home from my second Australian trip, my late colleague Prof. Joe Gillis (of Blechley Park fame) told me:

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.
- We did not manage to settle the conjecture, but worked jointly fruitfully on related problems, especially with Jamie Simpson.
- The group then got an extension of their grant and they again invited me, with the same type of results.
- After I returned home from my second Australian trip, my late colleague Prof. Joe Gillis (of Blechley Park fame) told me:
- "Aviezri, don't try too hard to prove your conjecture, otherwise you won't be invited again to Australia".

After 44 years, why hasn't the F-Conjecture been settled?

- In the Math Dept, Univ of Adelaide, Southern Australia, there once formed a research group that received a research grant to settle the F-Conjecture.
- They had a budget for inviting one foreign scientist, so, for some unexplained reason, they chose me.
- We did not manage to settle the conjecture, but worked jointly fruitfully on related problems, especially with Jamie Simpson.
- The group then got an extension of their grant and they again invited me, with the same type of results.
- After I returned home from my second Australian trip, my late colleague Prof. Joe Gillis (of Blechley Park fame) told me:
- "Aviezri, don't try too hard to prove your conjecture, otherwise you won't be invited again to Australia".

- Thank You Very Much