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# Splitting the Integers by Sequences

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- Notice that for both systems, the two largest moduli (2, 2 and 4, 4) are identical.
- Davenport, Mirsky, D. Newman, Radó proved, using a slick generating function and complex root of unity proof, that in any partitioning of  $\mathbb{Z}_{\geq 1}$  into  $m \geq 2$  arithmetic sequences, the two largest moduli are identical.

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- ... Erdos ... Berger, Felzenbaum, F. ...independently by Simpson.

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- Example: m = 3.

n	[7 <i>n</i> /4]	$\lfloor 7n/2 \rfloor - 1$	7 <i>n</i> – 3
1	1	2	4
2	3	6	
3	5		
4	7		

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- In other words, the only partitioning system by sequences into m ≥ 3 sets with distinct moduli is the indicated rational system!
- Google 'F Conjecture'.

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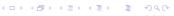
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- Example:  $\{\lfloor (2n)\alpha \rfloor\}_{n\geq 1}$ ,  $\{\lfloor (2n-1)\alpha \rfloor\}_{n\geq 1}$ ,  $\{\lfloor n\beta \rfloor\}_{n\geq 1}$  also partition  $\mathbb{Z}_{\geq 1}$ .

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- Of course also the multiplier n of  $\beta$  can be split into arbitrary arithmetic sequences.
- It follows from the result on partitioning with arithmetic sequences, that in any such irrational system with  $m \ge 3$ , two moduli (not necessarily the largest ones) are identical.



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but is wide open for the rationals.

 I find this to be the most tantalizing and fascinating aspect of the F-conjecture.

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- The F-Conjecture was proved by Simpson if the smallest modulus is at most 3/2, by Morikawa for m=3 and, under some condition, for m=4. Proofs in terms of balanced sequences have been given for m=3 by Tijdeman and for m=4 by Altman, Gaujal and Hordijk (unconditional). Later it was proved by Tijdeman for m=5 and 6, by Barát and Varjú for m=7.

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- Morikawa gave necessary and sufficient conditions for two rational sequences to be disjoint. Simpson simplified his difficult proof and dubbed it 'Japanese Remainder Theorem' in honor of Morikawa.

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- Toyota noticed that they spend a huge amount of resources in maintaining inventories of automotive parts, some of which become obsolete even before being called into use.
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- I think Tijdeman was the first to connect the F-Conjecture (partitioning numbers) with modern 'Just-In-Time' systems (partitioning time).

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- For 2,3 we could formulate game rules, but for m = 4 no game rules were found.

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